

# ON A FAMILY OF POLYNOMIALS RELATED TO $\zeta(2, 1) = \zeta(3)$

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**ABSTRACT.** We give a new proof of the identity  $\zeta(\{2, 1\}^l) = \zeta(\{3\}^l)$  of the multiple zeta values, where  $l = 1, 2, \dots$ , using generating functions of the underlying generalized polylogarithms. In the course of study we arrive at (hypergeometric) polynomials satisfying 3-term recurrence relations, whose properties we examine and compare with analogous ones of polynomials originated from an (ex-) conjectural identity of Borwein, Bradley and Broadhurst.

## 1. INTRODUCTION

The first thing one normally starts with, while learning about the multiple zeta values (MZVs)

$$\zeta(\mathbf{s}) = \zeta(s_1, s_2, \dots, s_l) = \sum_{n_1 > n_2 > \dots > n_l \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \dots n_l^{s_l}},$$

is Euler's identity  $\zeta(2, 1) = \zeta(3)$  — see [3] for an account of proofs and generalizations of the remarkable equality. One such generalization reads

$$\zeta(\{2, 1\}^l) = \zeta(\{3\}^l) \quad \text{for } l = 1, 2, \dots, \quad (1)$$

where the notation  $\{\mathbf{s}\}^m$  denotes the multi-index with  $m$  consecutive repetitions of the same index  $\mathbf{s}$ . The only known proof of (1) available in the literature makes use of the duality relation of MZVs, originally conjectured in [6] and shortly after established in [12]. The latter relation is based on a simple iterated-integral representation of MZVs (see [12] but also [3, 4, 14] for details) but, unfortunately, it is not capable of establishing similar-looking identities

$$\zeta(\{3, 1\}^l) = \frac{2\pi^{4l}}{(4l+2)!} \quad \text{for } l = 1, 2, \dots. \quad (2)$$

The equalities (2) were proven in [4] using a simple generating series argument.

The principal goal of this note is to give a proof of (1) via generating functions and to discuss, in this context, a related ex-conjecture of the alternating MZVs. An interesting outcome of this approach is a family of (hypergeometric) polynomials that satisfy a 3-term recurrence relation; a shape of the relation and (experimentally

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observed) structure of the zeroes of the polynomials suggest their bi-orthogonality origin [7, 8, 11].

## 2. MULTIPLE POLYLOGARITHMS

For  $l = 1, 2, \dots$ , consider the generalized polylogarithms

$$\begin{aligned} \text{Li}_{\{3\}^l}(z) &= \sum_{n_1 > n_2 > \dots > n_l \geq 1} \frac{z^{n_1}}{n_1^3 n_2^3 \dots n_l^3}, \\ \text{Li}_{\{2, 1\}^l}(z) &= \sum_{n_1 > m_1 > n_2 > m_2 > \dots > n_l > m_l \geq 1} \frac{z^{n_1}}{n_1^2 m_1 n_2^2 m_2 \dots n_l^2 m_l}, \\ \text{Li}_{\{\bar{2}, 1\}^l}(z) &= \sum_{n_1 > m_1 > n_2 > m_2 > \dots > n_l > m_l \geq 1} \frac{z^{n_1} (-1)^{n_1 + n_2 + \dots + n_l}}{n_1^2 m_1 n_2^2 m_2 \dots n_l^2 m_l}; \end{aligned}$$

if  $l = 0$  we set all these functions to be 1. Then at  $z = 1$ ,

$$\zeta(\{3\}^l) = \text{Li}_{\{3\}^l}(1) \quad \text{and} \quad \zeta(\{2, 1\}^l) = \text{Li}_{\{2, 1\}^l}(1),$$

and we also get the related alternating MZVs

$$\zeta(\{\bar{2}, 1\}^l) = \text{Li}_{\{\bar{2}, 1\}^l}(1)$$

from the specialization of the third polylogarithm.

Since

$$\begin{aligned} \left( (1-z) \frac{d}{dz} \right) \left( z \frac{d}{dz} \right)^2 \text{Li}_{\{3\}^l}(z) &= \text{Li}_{\{3\}^{l-1}}(z), \\ \left( (1-z) \frac{d}{dz} \right)^2 \left( z \frac{d}{dz} \right) \text{Li}_{\{2, 1\}^l}(z) &= \text{Li}_{\{2, 1\}^{l-1}}(z), \\ \left( (1+z) \frac{d}{dz} \right)^2 \left( z \frac{d}{dz} \right) \text{Li}_{\{\bar{2}, 1\}^l}(z) &= \text{Li}_{\{\bar{2}, 1\}^{l-1}}(-z) \end{aligned}$$

for  $l = 1, 2, \dots$ , the generating series

$$\begin{aligned} C(z; t) &= \sum_{l=0}^{\infty} \text{Li}_{\{3\}^l}(z) t^{3l}, \\ B(z; t) &= \sum_{l=0}^{\infty} \text{Li}_{\{2, 1\}^l}(z) t^{3l} \quad \text{and} \quad A(z; t) = \sum_{l=0}^{\infty} \text{Li}_{\{\bar{2}, 1\}^l}(z) t^{3l} \end{aligned}$$

satisfy linear differential equations. Namely, we have

$$\left( \left( (1-z) \frac{d}{dz} \right) \left( z \frac{d}{dz} \right)^2 - t^3 \right) C(z; t) = 0, \quad \left( \left( (1-z) \frac{d}{dz} \right) \left( z \frac{d}{dz} \right) - t^3 \right) B(z; t) = 0$$

and

$$\left( \left( (1-z) \frac{d}{dz} \right)^2 \left( z \frac{d}{dz} \right) \left( (1+z) \frac{d}{dz} \right)^2 \left( z \frac{d}{dz} \right) - t^6 \right) A(z; t) = 0,$$

respectively. The identities (1) and identities

$$\frac{1}{8^l} \zeta(\{2, 1\}^l) = \zeta(\{\overline{2}, 1\}^l) \quad \text{for } l = 1, 2, \dots,$$

conjectured in [4] and confirmed in [13] by means of a nice though sophisticated machinery of double shuffle relations and the ‘distribution’ relations (see also an outline in [2]), translate into

$$C(1; t) = B(1; t) = A(1; 2t).$$

Note that

$$C(1; t) = \sum_{l=0}^{\infty} t^{3l} \sum_{n_1 > n_2 > \dots > n_l \geq 1} \frac{1}{n_1^3 n_2^3 \dots n_l^3} = \prod_{j=1}^{\infty} \left(1 + \frac{t^3}{j^3}\right). \quad (3)$$

At the same time, the differential equation for  $C(z; t) = \sum_{n=0}^{\infty} C_n(t) z^n$  results in

$$-n^3 C_n + (n+1)^3 C_{n+1} = t^3 C_n$$

implying

$$\frac{C_{n+1}}{C_n} = \frac{n^3 + t^3}{(n+1)^3} = \frac{(n+t)(n+e^{2\pi i/3}t)(n+e^{4\pi i/3}t)}{(n+1)^3}$$

and leading to the hypergeometric form

$$C(z; t) = {}_3F_2 \left( \begin{matrix} t, \omega t, \omega^2 t \\ 1, 1 \end{matrix} \middle| z \right), \quad (4)$$

where  $\omega = e^{2\pi i/3}$ . We recall that

$${}_{m+1}F_m \left( \begin{matrix} a_0, a_1, \dots, a_m \\ b_1, \dots, b_m \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \dots (a_m)_n}{n! (b_1)_n \dots (b_m)_n} z^n,$$

where  $(a)_n = \Gamma(a+n)/\Gamma(a)$  denotes the Pochhammer symbol (also known as the ‘shifted factorial’ because  $(a)_n = a(a+1)\dots(a+n-1)$  for  $n = 1, 2, \dots$ ). It is not hard to see that the sequences  $A_n(t)$  and  $B_n(t)$  from  $A(z; t) = \sum_{n=0}^{\infty} A_n(t) z^n$  and  $B(z; t) = \sum_{n=0}^{\infty} B_n(t) z^n$  do not satisfy 2-term recurrence relations with polynomial coefficients. Thus, no hypergeometric representations of the type (4) are available for them.

### 3. SPECIAL POLYNOMIALS

The differential equation for  $B(z; t)$  translates into the 3-term recurrence relation

$$n^3 B_n - (n+1)^2 (2n+1) B_{n+1} + (n+2)^2 (n+1) B_{n+2} = t^3 B_n \quad (5)$$

for the coefficients  $B_n = B_n(t)$ ; the initial values are  $B_0 = 1$  and  $B_1 = 0$ .

**Lemma 1.** *We have*

$$B_n(t) = \frac{1}{n!} \sum_{k=0}^n \frac{(\omega t)_k (\omega^2 t)_k (t)_{n-k} (-t+k)_{n-k}}{k! (n-k)!} = \frac{(t)_n (-t)_n}{n!^2} {}_3F_2 \left( \begin{matrix} -n, \omega t, \omega^2 t \\ -t, 1-n-t \end{matrix} \middle| 1 \right). \quad (6)$$

*Proof.* The recursion (5) for the sequence in (6) follows from application of the Gosper–Zeilberger algorithm of creative telescoping. The initial values for  $n = 0$  and 1 are straightforward.  $\square$

**Remark.** The hypergeometric form in (6) was originally prompted by [9, Theorem 3.4]: the change of variable  $z \mapsto 1 - z$  in the differential equation for  $B(z; t)$  shows that the function  $f(z) = B(1 - z; t)$  satisfies the hypergeometric differential equation with upper parameters  $-t, -\omega t, -\omega^2 t$  and lower parameters  $0, 0$ .

It is not transparent from the formula (6) (but immediate from the recursion (5)) that  $B_n(t) \in t^3 \mathbb{Q}[t^3]$  for  $n = 0, 1, 2, \dots$ ; the classical transformations of  ${}_3F_2(1)$  and their representations as  ${}_6F_5(-1)$  hypergeometric series (see [1]) do not shed a light on this belonging either.

**Lemma 2.** *We have*

$$B(1; t) = \prod_{j=1}^{\infty} \left(1 + \frac{t^3}{j^3}\right). \quad (7)$$

*Proof.* This follows from the derivation

$$\begin{aligned} B(1; t) &= \sum_{n=0}^{\infty} B_n(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \frac{(\omega t)_k (\omega^2 t)_k (t)_{n-k} (-t+k)_{n-k}}{k! (n-k)!} \\ &= \sum_{k=0}^{\infty} \frac{(\omega t)_k (\omega^2 t)_k}{k!^2} \sum_{m=0}^{\infty} \frac{(t)_m (-t+k)_m}{m! (k+1)_m} \\ &= \sum_{k=0}^{\infty} \frac{(\omega t)_k (\omega^2 t)_k}{k!^2} \cdot {}_2F_1 \left( \begin{matrix} t, -t+k \\ k+1 \end{matrix} \middle| 1 \right) \\ &= \frac{1}{\Gamma(1-t)\Gamma(1+t)} \sum_{k=0}^{\infty} \frac{(\omega t)_k (\omega^2 t)_k}{k! (1-t)_k} \\ &= \frac{1}{\Gamma(1-t)\Gamma(1+t)} \cdot {}_2F_1 \left( \begin{matrix} \omega t, \omega^2 t \\ 1-t \end{matrix} \middle| 1 \right) \\ &= \frac{1}{\Gamma(1-t)\Gamma(1+t)} \cdot \frac{\Gamma(1-t)}{\Gamma(1-(1+\omega)t)\Gamma(1-(1+\omega^2)t)} \\ &= \frac{1}{\Gamma(1+t)\Gamma(1+\omega t)\Gamma(1+\omega^2 t)} = \prod_{j=1}^{\infty} \left(1 + \frac{t^3}{j^3}\right), \end{aligned}$$

where we applied twice Gauss's summation [1, Section 1.3]

$${}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| 1 \right) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)}$$

valid when  $\Re(c-a-b) > 0$ .  $\square$

Finally, we deduce from comparing (3) and (7),

**Theorem 1.** *The identity  $\zeta(\{3\}^l) = \zeta(\{2, 1\}^l)$  is valid for  $l = 1, 2, \dots$ .*

## 4. A GENERAL FAMILY OF POLYNOMIALS

It is not hard to extend Lemma 1 to the one-parameter family of polynomials

$$\begin{aligned} B_n^\alpha(t) &= \frac{1}{n!} \sum_{k=0}^n \frac{(\omega t)_k (\omega^2 t)_k (\alpha + t)_{n-k} (\alpha - t + k)_{n-k}}{k! (n-k)!} \\ &= \frac{1}{n!} \sum_{k=0}^n \frac{(\alpha + \omega t)_k (\alpha + \omega^2 t)_k (t)_{n-k} (\alpha - t + k)_{n-k}}{k! (n-k)!}. \end{aligned} \quad (8)$$

**Lemma 3.** *For each  $\alpha \in \mathbb{C}$ , the polynomials (8) satisfy the 3-term recurrence relation*

$$((n+\alpha)^3 - t^3) B_n^\alpha - (n+1)(2n^2 + 3n(\alpha+1) + \alpha^2 + 3\alpha + 1) B_{n+1}^\alpha + (n+2)^2 (n+1) B_{n+2}^\alpha = 0$$

and the initial conditions  $B_0^\alpha = 1$ ,  $B_1^\alpha = \alpha^2$ . In particular,  $B_n^\alpha(t) \in \mathbb{C}[t^3]$  for  $n = 0, 1, 2, \dots$ .

In addition, we have  $B_n^\alpha \in t^3 \mathbb{Q}[t^3]$  for  $\alpha = 0, -1, \dots, -n+1$  (in other words,  $B_n^\alpha(0) = 0$  for these values of  $\alpha$ ).

**Lemma 4.**  $B_n^{1-n-\alpha}(t) = B_n^\alpha(t)$ .

*Proof.* This follows from the hypergeometric representation

$$B_n^\alpha(t) = \frac{(\alpha + t)_n (\alpha - t)_n}{n!^2} {}_3F_2 \left( \begin{matrix} -n, \omega t, \omega^2 t \\ \alpha - t, 1 - \alpha - n - t \end{matrix} \middle| 1 \right). \quad \square$$

Here is one more property of the polynomials that follows from Euler's transformation [1, Section 1.2].

**Lemma 5.** *We have*

$$\sum_{n=0}^{\infty} B_n^\alpha(t) z^n = (1-z)^{1-2\alpha} \sum_{n=0}^{\infty} B_n^{1-\alpha}(t) z^n.$$

*Proof.* Indeed,

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^\alpha(t) z^n &= \sum_{k=0}^{\infty} \frac{(\omega t)_k (\omega^2 t)_k}{k!^2} z^k \cdot {}_2F_1 \left( \begin{matrix} \alpha + t, \alpha - t + k \\ k + 1 \end{matrix} \middle| z \right) \\ &= \sum_{k=0}^{\infty} \frac{(\omega t)_k (\omega^2 t)_k}{k!^2} z^k \cdot (1-z)^{1-2\alpha} {}_2F_1 \left( \begin{matrix} 1 - \alpha + t, 1 - \alpha - t + k \\ k + 1 \end{matrix} \middle| z \right) \\ &= (1-z)^{1-2\alpha} \sum_{n=0}^{\infty} B_n^{1-\alpha}(t) z^n. \end{aligned} \quad \square$$

*Alternative proof of Lemma 2.* It follows from Lemma 5 that

$$B_n^1(t) = \sum_{k=0}^n B_k(t),$$

hence  $B(1; t) = \lim_{n \rightarrow \infty} B_n^1(t)$  and the latter limit is straightforward from (8).  $\square$

Note that, with the help of the standard transformations of  ${}_3F_2(1)$  hypergeometric series, we can also write (8) as

$$B_n^\alpha(t) = \frac{(\alpha - \omega t)_n (\alpha - \omega^2 t)_n}{n!^2} {}_3F_2 \left( \begin{matrix} -n, \alpha + t, t \\ \alpha - \omega t, \alpha - \omega^2 t \end{matrix} \middle| 1 \right),$$

so that the generating functions of the continuous dual Hahn polynomials lead to the generating functions

$$\sum_{n=0}^{\infty} \frac{n!}{(\alpha - t)_n} B_n^\alpha(t) z^n = (1 - z)^{-t} {}_2F_1 \left( \begin{matrix} \alpha + \omega t, \alpha + \omega^2 t \\ \alpha - t \end{matrix} \middle| z \right)$$

and

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n n!}{(\alpha - \omega t)_n (\alpha - \omega^2 t)_n} B_n^\alpha(t) z^n = (1 - z)^{-\gamma} {}_3F_2 \left( \begin{matrix} \gamma, \alpha + t, t \\ \alpha - \omega t, \alpha - \omega^2 t \end{matrix} \middle| \frac{z}{z - 1} \right),$$

where  $\gamma$  is arbitrary.

Finally, numerical verification suggests that for real  $\alpha$  the zeroes of  $B_n^\alpha$  viewed as polynomials in  $x = t^3$  lie on the real half-line  $(-\infty, 0]$ .

## 5. POLYNOMIALS RELATED TO THE ALTERNATING MZV IDENTITY

Writing

$$\begin{aligned} A(z; t) &= \sum_{n=0}^{\infty} A_n(t) z^n \\ &= 1 + \frac{1}{4} t^3 z^2 - \frac{1}{6} t^3 z^3 + \left( \frac{1}{192} t^3 + \frac{11}{96} \right) t^3 z^4 - \left( \frac{1}{240} t^3 + \frac{1}{12} \right) t^3 z^5 \\ &\quad + \left( \frac{1}{34560} t^6 + \frac{23}{5760} t^3 + \frac{137}{2160} \right) t^3 z^6 + O(z^7) \end{aligned}$$

and using the equation

$$\left( (1 + z) \frac{d}{dz} \right)^2 \left( z \frac{d}{dz} \right) A(z; t) = t^3 A(-z; t),$$

we deduce that

$$(n^3 - T) A_n + (n + 1)^2 (2n + 1) A_{n+1} + (n + 2)^2 (n + 1) A_{n+2} = 0, \quad (9)$$

where  $T = (-1)^n t^3$ . Producing two shifted copies of (9),

$$((n - 1)^3 + T) A_{n-1} + n^2 (2n - 1) A_n + (n + 1)^2 n A_{n+1} = 0, \quad (10)$$

$$((n - 2)^3 - T) A_{n-2} + (n - 1)^2 (2n - 3) A_{n-1} + n^2 (n - 1) A_n = 0, \quad (11)$$

then multiplying recursion (9) by  $n(n - 1)^2 (2n - 3)$ , recursion (10) by  $-(n - 1)^2 \times (2n + 1)(2n - 3)$ , recursion (11) by  $(2n + 1)((n - 1)^3 + T)$  and adding the three equations so obtained we arrive at

$$\begin{aligned} &(2n + 1)((n - 1)^3 + T)((n - 2)^3 - T) A_{n-2} \\ &\quad - n(n - 1)(2n - 1)(2n(n - 1)(n^2 - n - 1) - 3T) A_n \\ &\quad + (n + 2)^2 (n + 1) n(n - 1)^2 (2n - 3) A_{n+2} = 0. \end{aligned}$$

This final recursion restricted to the subsequence  $A_{2n}$ , namely

$$\begin{aligned} & (4n+5)((2n)^3 - t^3)((2n+1)^3 + t^3)A_{2n} \\ & - (4n+3)(2n+1)(2n+2)(2(2n+1)(2n+2)(4n^2+6n+1) - 3t^3)A_{2n+2} \\ & + (4n+1)(2n+1)^2(2n+2)(2n+3)(2n+4)^2A_{2n+4} = 0, \end{aligned} \quad (12)$$

and, similarly, to  $A_{2n+1}$  gives rise to two families of so-called Frobenius–Stickelberger–Thiele polynomials [10]. The latter connection, however, sheds no light on the asymptotics of  $A_n(t) \in \mathbb{Q}[t^3]$ . Unlike the case of  $B(z; t)$  treated in Section 3 we cannot find closed form expressions for those subsequences. Here is the case most visually related to the recursion (12):

$$\begin{aligned} & (4n+5)\frac{(2n)^3 - t^3}{t-2n}\frac{(2n+1)^3 + t^3}{t+2n+1}A'_n \\ & - (4n+3)(2n+1)(2n+2)((2n+1)(2n+2) + (8n^2+12n+1)t + 3t^2)A'_{n+1} \\ & + (4n+1)(2n+1)^2(2n+2)(2n+3)(2n+4)^2A'_{n+2} = 0, \end{aligned}$$

where

$$A'_n = \frac{1}{2^n (1/2)_n n!} \sum_{k=0}^n \frac{(\omega t/2)_k (\omega^2 t/2)_k (t/2)_{n-k} (1/2)_{n-k}}{k! (n-k)!} (-1)^k.$$

The latter polynomials are not from  $\mathbb{Q}[t^3]$ .

If we consider  $\tilde{A}_n(t) = \sum_{k=0}^n A_k(t)$  then (it is already known [5, 13] that)

$$(n^3 - (-1)^n t^3)\tilde{A}_{n-1} + (2n+1)n\tilde{A}_n - (n+1)^2 n\tilde{A}_{n+1} = 0, \quad n = 1, 2, \dots$$

As before, the standard elimination translates it into

$$\begin{aligned} & (2n+3)((n-1)^3 + T)(n^3 - T)\tilde{A}_{n-2} \\ & - (2n+1)n(n-1)(2(n^2+n+1)^2 - 6 - T)\tilde{A}_n \\ & + (2n-1)(n+2)^2(n+1)^2 n(n-1)\tilde{A}_{n+2} = 0, \end{aligned}$$

where  $T = (-1)^n t^3$ . One can easily verify that

$$\tilde{A}_n(t) = 1 + \dots + \frac{t^{3\lfloor n/2 \rfloor}}{2^{\lfloor n/2 \rfloor} \lfloor n/2 \rfloor! n!}$$

but we also lack an explicit representation for them.

We have checked numerically a fine behaviour (orthogonal-polynomial-like) of the zeroes of  $A_n$  and  $\tilde{A}_n$  viewed as polynomials in  $x = t^3$  (both of degree  $\lfloor n/2 \rfloor$  in  $x$ ). Namely, all the zeroes lie on the real half-line  $(-\infty, 0]$ . This is in line with the property of the polynomials  $B_n^\alpha$  (see the last paragraph in Section 4).

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